

On Newman's phenomenon in higher bases

Sai Teja Somu

Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee - 247 667, India
somuteja@gmail.com

Abstract

A well known result of Newman says that upto a limit, multiples of 3 with even number of 1's in binary representation always exceed multiples of 3 with odd number of 1's. The phenomenon of preponderance of even number of 1's is now known as Newman's phenomenon. We show that this phenomenon exists for higher bases. Let b be a positive integer (≥ 2). Let A_b be the set of all natural numbers which contain only 0's and 1's in b -ary expansion and $S_{q,i}^{(b)}(n)$ be the difference between the corresponding number of $k_e < n$, $k_e \equiv i \pmod q$, $k_e \in A_b$ and k_e has even number of 1's in b -ary expansion and the number of $k_o < n$, $k_o \equiv i \pmod q$, $k_o \in A_b$ and k_o has odd number of 1's in b -ary expansion. Let q be a multiple or divisor of $b+1$ which is relatively prime to b then we show that $S_{q,0}^{(b)}(n) > 0$ for sufficiently large n . We show that there is a stronger Newman's phenomenon in A_b in the following sense. If $b > 2$ and $n = \sum_{i=0}^{k-1} b_i 2^i$ with $b_i \in \{0, 1\}$, let $b(n) = \sum_{i=0}^{k-1} b_i b^i$ then $\lim_{n \rightarrow \infty} \frac{S_{3,0}^{(2)}(n)}{S_{b+1,0}^{(b)}(b(n))} = 0$. That is, for the same number of terms there is stronger preponderance in A_b than in $A_2 = \mathbb{N}$. In the last section we show that number of primes $p \leq x$ for which $S_{p,0}^{(b)}(n) > 0$ for sufficiently large n is $o\left(\frac{x}{\log x}\right)$.

1 Introduction

L.Moser conjectured that when the multiples of 3 are written in binary, upto a limit, numbers with even number of 1's always exceed numbers with odd number of 1's. Newman in [7] proved that the conjecture is true. To be precise, he proved that $S_{3,0}^{(2)}(n) > 0$ for all n . If we consider a number n let $b_{k-1} \cdots b_0$ be the binary expansion of n . Since $2 \equiv -1 \pmod 3$, we have

$$n \equiv b_0 - b_1 + b_2 \cdots + (-1)^{k-1} b_{k-1} \pmod 3.$$

Let N be a natural number and $N = n_{k-1} n_{k-2} \cdots n_0$ be binary expansion of N . The difference of multiples of 3 having even number of 1's and odd number of 1's is

$$S_{3,0}^{(2)}(n) = \sum_{\substack{b_0 - b_1 + \cdots + (-1)^{k-1} b_{k-1} \equiv 0 \pmod 3 \\ b_{k-1} b_{k-2} \cdots b_0 < n_{k-1} n_{k-2} \cdots n_0}} (-1)^{b_0 + \cdots + b_{k-1}}, \quad (1.1)$$

where $<$ denotes lexicographic order among sequences of finite length. In (1) if we replace 2 by an even natural number b then the sum appears to increase as b increases. That is, if we

consider the following sum

$$S_{b+1,0}^{(b)}(b(n)) = \sum_{\substack{b_0 - b_1 + \dots + (-1)^{k-1} b_{k-1} \equiv 0 \pmod{b} \\ b_{k-1} b_{k-2} \dots b_0 < n_{k-1} n_{k-2} \dots n_0}} (-1)^{b_0 + \dots + b_{k-1}} \quad (1.2)$$

then the sum seems to increase as b increases. So we can guess that $S_{b+1,0}^{(b)}(b(n)) > 0$ for sufficiently large n . We prove this result in section 2. Let A_b be the set of all natural numbers which contain only 0's and 1's in b -ary expansion. Then upto a limit, multiples of $b+1$ with even number of 1's in A_b always exceed multiples of $b+1$ with odd number of 1's in A_b . So there is Newman's phenomenon among the multiples of $b+1$ in the set A_b . In fact the Newman's phenomenon gets stronger as b increases in the following sense. We show that for any two even numbers b_1, b_2 with $b_1 > b_2$ we have

$$\lim_{n \rightarrow \infty} \frac{S_{b_2+1,0}^{(b_2)}(b_2(n))}{S_{b_1+1,0}^{(b_1)}(b_1(n))} = 0.$$

That is, for the same number of terms in the sets A_{b_1} and A_{b_2} there is a stronger preponderance in A_{b_1} compared to A_{b_2} .

In fact we prove more than $S_{b+1,0}^{(b)}(b(n)) > 0$. We prove the following theorem in section 2.

Theorem 1.1. *If b and q are two positive integers such that $b \geq 2$, $(b+1)|q$ and $(b, q) = 1$ where (b, q) is the greatest common divisor of q and b . Let v be an integer, then for sufficiently large N*

1. $S_{q,v(b+1)}^{(b)}(N) > 0$.
2. $S_{q,v(b+1)+1}^{(b)}(N) < 0$.
3. if $b \leq 3$ then $S_{q,v(b+1)-1}^{(b)}(N) < 0$.

Theorem 1.1 partially generalizes Theorem 1 of [2] which states that $S_{3k,0}^{(2)}(n) > 0$ for almost all n .

In section 3 we prove the following result.

Theorem 1.2. *If $d > 1$ is a divisor of $(b+1)$ for $b \geq 2$, then for sufficiently large N*

1. $S_{d,0}^{(b)}(N) > 0$.
2. $S_{d,1}^{(b)}(N) < 0$.
3. $S_{d,-1}^{(b)}(N) \leq 0$ and if $d > 3$ then $S_{d,-1}^{(b)}(N) < 0$.

Let $\mathbb{P}_t(b)$ denote the set of all primes such that $\frac{p-1}{\text{ord}_b(p)} = t$ where $\text{ord}_p(b)$ denotes the order of b in the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^*$. In the last section we prove that the primes satisfying $S_{p,0}^{(b)}(N) > 0$ are of zero density in the set of primes.

Theorem 1.3. *The number of primes $p \in \mathbb{P}_t(b)$ for a given $t > 1$ such that $S_{p,0}^{(b)}(N) > 0$ for sufficiently large N are bounded by*

$$p \leq C^* t^2 \log^2 t,$$

where $C^* > 0$ only depends on b . If $t = 1$ the number of primes are bounded by

$$p \leq C_1,$$

where $C_1 > 0$ only depends on b . Furthermore, the total number of primes $p \leq x$ such that $S_{p,0}^{(b)}(n) > 0$ for sufficiently large n is $o\left(\frac{x}{\log x}\right)$ as $x \rightarrow \infty$.

Theorem 1.3 is a generalization of Theorem 2 of [2].

2 Proof of Theorem 1.1

If b is an odd number as q is divisible by $b+1$, q will be an even integer and all the integers satisfying the congruences $n \equiv v(b+1) \pmod{q}$, $n \equiv v(b+1)+1 \pmod{q}$ and $n \equiv v(b+1)-1 \pmod{q}$ are even, odd and odd respectively. Also $n \equiv s_b(n) \pmod{2}$, where $s_b(n)$ denotes sum of digits in b -ary expansion. Hence all the numbers satisfying the congruences $n \equiv v(b+1) \pmod{q}$, $n \equiv v(b+1)+1 \pmod{q}$ and $n \equiv v(b+1)-1 \pmod{q}$ will have even b -ary digit sum, odd b -ary digit sum and odd b -ary digit sum respectively so (1),(2) and (3) of Theorem 1.1 are trivially true when $b > 1$ is an odd number. Hence we can assume that b is even. We prove six lemmas in order to prove Theorem 1.1.

Lemma 2.1. *If $N < b^k$,*

$$S_{q,i}^{(b)}(b^k + N) = S_{q,i}^{(b)}(b^k) - S_{q,i-b^k}^{(b)}(N).$$

Proof. Note that if $n < b^k$ then $s_b(n + b^k) = s_b(n) + 1$. Thus, we have

$$\begin{aligned} S_{q,i}^{(b)}(b^k + n) &= \sum_{\substack{n \in A_b \\ 0 \leq n < N+b^k \\ n \equiv i \pmod{q}}} (-1)^{s_b(n)} \\ &= \sum_{\substack{n \in A_b \\ 0 \leq n < b^k \\ n \equiv i \pmod{q}}} (-1)^{s_b(n)} + \sum_{\substack{n \in A_b \\ 0 \leq n < N \\ n \equiv i-b^k \pmod{q}}} (-1)^{s_b(n+b^k)} \\ &= \sum_{\substack{n \in A_b \\ 0 \leq n < b^k \\ n \equiv i \pmod{q}}} (-1)^{s_b(n)} - \sum_{\substack{n \in A_b \\ 0 \leq n < N \\ n \equiv i-b^k \pmod{q}}} (-1)^{s_b(n)} \\ &= S_{q,i}^{(b)}(b^k) - S_{q,i-b^k}^{(b)}(N). \end{aligned}$$

□

From Lemma 2.1 it follows that if $N = b^{k_1} + \dots + b^{k_r}$ and $b^{k_1} > \dots > b^{k_r}$ then

$$S_{q,i}^{(b)}(N) = S_{q,i}^{(b)}(b^{k_1}) - S_{q,i-b^{k_1}}^{(b)}(b^{k_2}) - \dots + (-1)^{r-1} S_{q,i-b^{k_1}-\dots-b^{k_{r-1}}}^{(b)}(-1, b^{k_r}). \quad (2.3)$$

Lemma 2.2. *If k is a natural number then*

$$S_{q,i}^{(b)}(b^k) = \frac{1}{q} \sum_{m=0}^{q-1} \zeta_q^{-im} \prod_{p=0}^{k-1} (1 - \zeta_q^{mb^p}).$$

Proof. If k is a natural number, we have

$$\begin{aligned} S_{q,i}^{(b)}(b^k) &= \sum_{\substack{n \in A_b \\ 0 \leq n < b^k \\ n \equiv i \pmod{q}}} (-1)^{s_b(n)} \\ &= \frac{1}{q} \sum_{\substack{n \in A_b \\ 0 \leq n < b^k}} \sum_{m=0}^{q-1} (-1)^{s_b(n)} \zeta_q^{(n-i)m} \\ &= \frac{1}{q} \sum_{0 \leq \epsilon_0, \dots, \epsilon_{k-1} \leq 1} \sum_{m=0}^{q-1} (-1)^{\epsilon_0 + \dots + \epsilon_{k-1}} \zeta_q^{(\epsilon_0 + \epsilon_1 b + \dots + \epsilon_{k-1} b^{k-1} - i)m} \\ &= \frac{1}{q} \sum_{m=0}^{q-1} \zeta_q^{-im} \prod_{p=0}^{k-1} (1 - \zeta_q^{mb^p}). \end{aligned}$$

□

Lemma 2.3. *Let $b = 2l$. If $|z| = 1$ and $|1 - z| > 2 \sin \frac{\pi l}{2l+1}$, then*

$$|1 - z||1 - z^b| < 4 \sin^2 \frac{\pi l}{2l+1}.$$

Proof. Let z be $e^{i\theta}$, where $\theta \in [0, 2\pi]$. As $|1 - z| > 2 \sin \frac{\pi l}{2l+1}$ we have $\frac{2\pi l}{2l+1} < \theta < \frac{2\pi(l+1)}{2l+1}$. Let

$$f(\theta) := 4 \left| \sin l\theta \sin \frac{\theta}{2} \right| = |1 - z||1 - z^b|.$$

If $\frac{2\pi l}{2l+1} < \theta < \frac{2\pi(l+1)}{2l+1}$, it can be seen that $|\tan l\theta| \leq |\tan \frac{\pi l}{2l+1}| \leq |\tan \frac{\theta}{2}|$.

We have

$$\begin{aligned} f(\theta) &= 4\delta \sin l\theta \sin \frac{\theta}{2}, \\ f'(\theta) &= 4\delta \left(l \cos l\theta \sin \frac{\theta}{2} + \frac{1}{2} \sin l\theta \cos \frac{\theta}{2} \right). \end{aligned}$$

	θ	$\sin l\theta$	$\cos l\theta$	$\sin \frac{\theta}{2}$	$\cos \frac{\theta}{2}$	δ	$f'(\theta)$
$l = 2m$	$\frac{2l\pi}{2l+1} < \theta \leq \pi$	≤ 0	≥ 0	> 0	> 0	-1	< 0
$l = 2m$	$\pi < \theta < \frac{(2l+2)\pi}{2l+1}$	> 0	> 0	> 0	< 0	$+1$	> 0
$l = 2m + 1$	$\frac{2l\pi}{2l+1} < \theta \leq \pi$	> 0	< 0	> 0	> 0	$+1$	< 0
$l = 2m + 1$	$\pi < \theta < \frac{(2l+2)\pi}{2l+1}$	< 0	< 0	> 0	< 0	-1	> 0

So, in any interval

$$f(\theta) < f(\pi \pm \frac{\pi}{2l+1}) = 4 \sin^2 \frac{\pi l}{2l+1}.$$

□

Let s be order of b in the multiplicative group $(\mathbb{Z}/q\mathbb{Z})^*$.

Lemma 2.4. *If $(q, b) = 1$, $(b+1)|q$ and b is even, $m \equiv \frac{\pm ql}{2l+1} \pmod{q}$ then*

$$\left| \prod_{p=0}^{s-1} (1 - \zeta_q^{b^p m}) \right| = \left(2 \sin \frac{\pi l}{2l+1} \right)^s,$$

and if $m \not\equiv \frac{\pm ql}{2l+1} \pmod{q}$ then

$$\left| \prod_{p=0}^{s-1} (1 - \zeta_q^{b^p m}) \right| < \left(2 \sin \frac{\pi l}{2l+1} \right)^s.$$

Proof. Observe that $|1 - \zeta_q^{mb^p}| = 2 \sin \frac{\pi l}{2l+1}$ for $0 \leq p \leq s-1$ if and only if $m \equiv \frac{\pm ql}{2l+1} \pmod{q}$. Hence

$$\left| \prod_{p=0}^{s-1} (1 - \zeta_q^{b^p m}) \right| = \left(2 \sin \frac{\pi l}{2l+1} \right)^s$$

when $m \equiv \frac{\pm ql}{2l+1} \pmod{q}$. If $m \not\equiv \frac{\pm ql}{2l+1} \pmod{q}$ then $|1 - \zeta_q^{mb^i}| \neq 2 \sin \frac{\pi l}{2l+1}$ for $0 \leq i \leq s-1$. Let S_1, S_2 and S_3 be subsets of $\{0, 1, \dots, s-1\}$. S_1 contains all the p such that $|1 - \zeta_q^{mb^p}| > 2 \sin \frac{\pi l}{2l+1}$, S_2 all p such that $p-1 \pmod{s} \in S_1$ and S_3 contains the remaining elements of $\{0, 1, \dots, s-1\}$. Clearly, from Lemma 2.3 $S_1 \cap S_2 = \emptyset$ and if $p \in S_1$ then $|(1 - \zeta_q^{b^p m})(1 - \zeta_q^{b^{p+1} m})| < (2 \sin \frac{\pi l}{2l+1})^2$. Therefore

$$\begin{aligned} \left| \prod_{p=0}^{s-1} (1 - \zeta_q^{b^p m}) \right| &= \prod_{p \in S_1} |(1 - \zeta_q^{b^p m})(1 - \zeta_q^{b^{p+1} m})| \prod_{p \in S_3} |1 - \zeta_q^{b^p m}| \\ &< \left(2 \sin \frac{\pi l}{2l+1} \right)^s. \end{aligned}$$

□

Let $\gamma = \max \left\{ \left| \prod_{p=0}^{s-1} (1 - \zeta_q^{b^p m}) \right| : m \not\equiv \pm \frac{ql}{2l+1} \pmod{q} \right\}$. From Lemma 2.4 we have $\gamma < \left(2 \sin \frac{\pi l}{2l+1} \right)^s$.

Lemma 2.5.

$$S_{q,i}^{(b)}(b^k) = \begin{cases} \frac{2}{q} \left(\cos \frac{2\pi il}{2l+1} \right) \left(2 \sin \frac{\pi l}{2l+1} \right)^k + O(\gamma^{\frac{k}{s}}) & k \text{ even} \\ \frac{2}{q} \left(\cos \frac{2\pi il}{2l+1} - \cos \frac{2\pi(i-1)l}{2l+1} \right) \left(2 \sin \frac{\pi l}{2l+1} \right)^{k-1} + O(\gamma^{\frac{k}{s}}) & k \text{ odd.} \end{cases}$$

Proof. Let $k = k_1 s + k_2$ where $0 \leq k_2 \leq s-1$. From Lemma 2.2 we have

$$S_{q,i}^{(b)}(b^k) = \frac{1}{q} \sum_{m \equiv \frac{\pm ql}{2l+1}} \zeta_q^{-im} \prod_{p=0}^{k-1} (1 - \zeta_q^{b^p m}) + \frac{1}{q} \sum_{m \not\equiv \frac{\pm ql}{2l+1}} \zeta_q^{-im} \prod_{p=0}^{k-1} (1 - \zeta_q^{b^p m}) \quad (2.4)$$

where the first term of right hand side of equation (2.4) is

$$\frac{1}{q} \sum_{m \equiv \frac{\pm ql}{2l+1}} \zeta_q^{-im} \prod_{p=0}^{k-1} (1 - \zeta_q^{b^p m}) = \frac{1}{q} \zeta_{b+1}^{-il} \prod_{p=0}^{k-1} (1 - \zeta_{b+1}^{lb^p}) + \frac{1}{q} \zeta_{b+1}^{+il} \prod_{p=0}^{k-1} (1 - \zeta_{b+1}^{-lb^p}).$$

Hence

$$\frac{1}{q} \sum_{m \equiv \frac{\pm ql}{2l+1}} \zeta_q^{-im} \prod_{p=0}^{k-1} (1 - \zeta_q^{b^p m}) = \begin{cases} \frac{2}{q} \cos \frac{2\pi il}{2l+1} \left(2 \sin \frac{\pi l}{2l+1} \right)^k & k \text{ even} \\ \frac{2}{q} \left(\cos \frac{2\pi il}{2l+1} - \cos \frac{2\pi(i-1)l}{2l+1} \right) \left(2 \sin \frac{\pi l}{2l+1} \right)^{k-1} & k \text{ odd.} \end{cases}$$

The second term of right hand side of equation (2.4)

$$\left| \frac{1}{q} \sum_{m \not\equiv \frac{\pm ql}{2l+1}} \zeta_q^{-im} \prod_{p=0}^{k-1} (1 - \zeta_q^{b^p m}) \right| = \left| \frac{1}{q} \sum_{m \not\equiv \frac{\pm ql}{2l+1}} \zeta_q^{-im} \left(\prod_{p=0}^{s-1} (1 - \zeta_q^{b^p m}) \right)^{k_1} \left(\prod_{p=0}^{k_2-1} (1 - \zeta_q^{b^p m}) \right) \right|.$$

Let $C = \max\left\{ \left| \prod_{p=0}^{k_2-1} (1 - \zeta_q^{b^p m}) \right| : m \in \mathbb{N}, 0 \leq k_2 \leq s-1 \right\}$ then we have

$$\left| \frac{1}{q} \sum_{m \not\equiv \frac{\pm ql}{2l+1}} \zeta_q^{-im} \prod_{p=0}^{k-1} (1 - \zeta_q^{b^p m}) \right| \leq C \gamma^{k_1} = O(\gamma^{\frac{k}{s}})$$

which implies

$$S_{q,i}^{(b)}(b^k) = \begin{cases} \frac{2}{q} \cos \frac{2\pi il}{2l+1} \left(2 \sin \frac{\pi l}{2l+1} \right)^k + O(\gamma^{\frac{k}{s}}), & k \text{ even} \\ \frac{2}{q} \left(\cos \frac{2\pi il}{2l+1} - \cos \frac{2\pi(i-1)l}{2l+1} \right) \left(2 \sin \frac{\pi l}{2l+1} \right)^{k-1} + O(\gamma^{\frac{k}{s}}) & k \text{ odd.} \end{cases}$$

□

Lemma 2.6. If $b \geq 4$ is even and $b = 2l$ then there exists a constant $M > 0$ depending upon b, q and not depending on k such that

1.

$$S_{q,v(b+1)}^{(b)}(b^k) \geq \frac{1}{q} \left(2 \sin \frac{\pi l}{2l+1} \right)^{k+1} - M\gamma^{\frac{k}{s}}$$

2.

$$S_{q,v(b+1)+1}^{(b)}(b^k) \leq \frac{2}{q} \cos \frac{2\pi l}{2l+1} \left(2 \sin \frac{\pi l}{2l+1} \right)^k + M\gamma^{\frac{k}{s}}$$

3.

$$S_{q,v(b+1)-1}^{(b)}(b^k) \leq \frac{2}{q} \left(\cos \frac{2\pi l}{2l+1} - \cos \frac{4\pi l}{2l+1} \right) \left(2 \sin \frac{\pi l}{2l+1} \right)^{k-1} + M\gamma^{\frac{k}{s}}$$

4.

$$\left| S_{q,i}^{(b)}(b^k) \right| \leq \frac{2}{q} \left(2 \sin \frac{\pi l}{2l+1} \right)^k + M\gamma^{\frac{k}{s}}$$

5.

$$S_{q,v(b+1)\pm 1}^{(b)}(b^k) \leq M\gamma^{\frac{k}{s}}$$

6.

$$S_{q,v(b+1)+0 \text{ or } \pm 2}^{(b)}(b^k) \geq -M\gamma^{\frac{k}{s}}$$

Proof. From Lemma 2.5 we have

$$S_{q,i}^{(b)}(b^k) = \begin{cases} \frac{2}{q} \left(\cos \frac{2\pi i l}{2l+1} \right) \left(2 \sin \frac{\pi l}{2l+1} \right)^k + O(\gamma^{\frac{k}{s}}) & k \text{ even} \\ \frac{2}{q} \left(\cos \frac{2\pi i l}{2l+1} - \cos \frac{2\pi(i-1)l}{2l+1} \right) \left(2 \sin \frac{\pi l}{2l+1} \right)^{k-1} + O(\gamma^{\frac{k}{s}}) & k \text{ odd.} \end{cases}$$

Hence for $i = v(b+1)$, $v(b+1)+1$ and $v(b+1)-1$

$$\begin{aligned} S_{q,v(b+1)}^{(b)}(b^k) &= \begin{cases} \frac{2}{q} \left(2 \sin \frac{\pi l}{2l+1} \right)^k + O(\gamma^{\frac{k}{s}}) & k \text{ even} \\ \frac{1}{q} \left(2 \sin \frac{\pi l}{2l+1} \right)^{k+1} + O(\gamma^{\frac{k}{s}}) & k \text{ odd} \end{cases} \\ S_{q,v(b+1)+1}^{(b)}(b^k) &= \begin{cases} \frac{2}{q} \cos \frac{2\pi l}{2l+1} \left(2 \sin \frac{\pi l}{2l+1} \right)^k + O(\gamma^{\frac{k}{s}}) & k \text{ even} \\ -\frac{1}{q} \left(2 \sin \frac{\pi l}{2l+1} \right)^{k+1} + O(\gamma^{\frac{k}{s}}) & k \text{ odd} \end{cases} \\ S_{q,v(b+1)-1}^{(b)}(b^k) &= \begin{cases} \frac{2}{q} \cos \frac{2\pi l}{2l+1} \left(2 \sin \frac{\pi l}{2l+1} \right)^k + O(\gamma^{\frac{k}{s}}) & k \text{ even} \\ \frac{2}{q} \left(\cos \frac{2\pi l}{2l+1} - \cos \frac{4\pi l}{2l+1} \right) \left(2 \sin \frac{\pi l}{2l+1} \right)^{k-1} + O(\gamma^{\frac{k}{s}}) & k \text{ odd.} \end{cases} \end{aligned}$$

Results (1),(2),(3) and (4) follow from the inequalities

$$\begin{aligned}
2 \sin \frac{\pi l}{2l+1} &\leq 2, \\
2 \cos \frac{2\pi l}{2l+1} &\geq -2 \sin \frac{\pi l}{2l+1}, \\
\cos \frac{2\pi l}{2l+1} - \cos \frac{4\pi l}{2l+1} &\geq 2 \cos \frac{2\pi l}{2l+1} \sin \frac{\pi l}{2l+1}, \\
\left| \cos \frac{2\pi l}{2l+1} \right| &\leq 1 \text{ and} \\
\left| \cos \frac{2\pi i l}{2l+1} - \cos \frac{2\pi(i-1)l}{2l+1} \right| &\leq 2 \sin \frac{\pi l}{2l+1}.
\end{aligned}$$

If $l \geq 2$ from the inequalities $\cos \frac{2\pi l}{2l+1} < 0$ and $\cos \frac{4\pi l}{2l+1} > 0$ we have

$$S_{q,v(b+1)\pm 1}^{(b)}(b^k) \leq M\gamma^{\frac{k}{s}},$$

for some M . Hence (5) is true.

$$S_{q,v(b+1)+2}^{(b)}(b^k) = \begin{cases} \frac{2}{q} \cos \frac{4\pi l}{2l+1} \left(2 \sin \frac{\pi l}{2l+1}\right)^k + O(\gamma^{\frac{k}{s}}) & k \text{ even} \\ \frac{2}{q} \left(\cos \frac{4\pi l}{2l+1} - \cos \frac{2\pi l}{2l+1}\right) \left(2 \sin \frac{\pi l}{2l+1}\right)^{k-1} + O(\gamma^{\frac{k}{s}}) & k \text{ odd} \end{cases}$$

Hence

$$\begin{aligned}
S_{q,v(b+1)+2}^{(b)}(b^k) &\geq -M\gamma^{\frac{k}{s}}. \\
S_{q,v(b+1)-2}^{(b)}(b^k) &= \begin{cases} \frac{2}{q} \cos \frac{4\pi l}{2l+1} \left(2 \sin \frac{\pi l}{2l+1}\right)^k + O(\gamma^{\frac{k}{s}}) & k \text{ even} \\ \frac{2}{q} \left(2 \sin \frac{\pi l}{2l+1}\right)^{k-1} \left(\cos \frac{4\pi l}{2l+1} - \cos \frac{6\pi l}{2l+1}\right) + O(\gamma^{\frac{k}{s}}) & k \text{ odd} \end{cases}
\end{aligned}$$

As $\cos \frac{4\pi l}{2l+1} - \cos \frac{6\pi l}{2l+1} \geq 0$ for $l \geq 2$ we have

$$S_{q,v(b+1)-2}^{(b)}(b^k) \geq -M\gamma^{\frac{k}{s}}$$

implying (6). □

Proof of Theorem 1.1

Proof. Theorem 1.3 of [3] and Theorem 1 of [2] covers the case $b = 2$ so we can assume $b \geq 4$. From (2.3) if

$$N = b^{k_1} + \dots + b^{k_r}$$

where $k_1 > \dots > k_r$ we have

$$S_{q,i}^{(b)}(N) = S_{q,i}^{(b)}(b^{k_1}) - S_{q,i-b^{k_1}}^{(b)}(b^{k_2}) + S_{q,i-b^{k_1}-b^{k_2}}^{(b)}(b^{k_3}) - \dots$$

When $i = v(b+1)$ we have

$$S_{q,v(b+1)}^{(b)}(N) = S_{q,v(b+1)}^{(b)}(b^{k_1}) - S_{q,v(b+1)-b^{k_1}}^{(b)}(b^{k_2}) + S_{q,v(b+1)-b^{k_1}-b^{k_2}}^{(b)}(b^{k_3}) - \dots$$

As $b^k \equiv \pm 1 \pmod{b+1}$, from Lemma 2.6 we have

$$\begin{aligned} S_{q,v(b+1)}^{(b)}(b^k) &\geq \frac{1}{q} \left(2 \sin \frac{\pi l}{2l+1} \right)^{k+1} - M \gamma^{\frac{k}{s}}, \\ S_{q,v(b+1)-b^{k_1}}^{(b)}(b^{k_2}) &\leq M \gamma^{\frac{k}{s}} \text{ and} \\ \left| S_{q,i}^{(b)}(b^k) \right| &\leq \frac{2}{q} \left(2 \sin \frac{\pi l}{2l+1} \right)^k + M \gamma^{\frac{k}{s}}. \end{aligned}$$

Let $\beta = 2 \sin \frac{\pi l}{2l+1}$ then

$$\begin{aligned} S_{q,v(b+1)}^{(b)}(N) &\geq (1 + o(1)) \left(\frac{1}{q} \left(2 \sin \frac{\pi l}{2l+1} \right)^{k_1+1} - \frac{2}{q} \left(2 \sin \frac{\pi l}{2l+1} \right)^{k_1-2} - \frac{2}{q} \left(2 \sin \frac{\pi l}{2l+1} \right)^{k_1-3} + \dots \right) \\ &= (1 + o(1)) \left(\frac{\beta^{k_1+1}}{q} - \frac{2 \beta^{k_1-1}}{q \beta - 1} \right). \end{aligned}$$

Hence (1) of Theorem 1.1 is true.

$$S_{q,v(b+1)+1}^{(b)}(N) = S_{q,v(b+1)+1}^{(b)}(b^{k_1}) - S_{q,v(b+1)+1-b^{k_1}}^{(b)}(b^{k_2}) + S_{q,v(b+1)+1-b^{k_1}-b^{k_2}}^{(b)}(b^{k_3}) - \dots$$

From (2) of Lemma 2.6 and from (6) of Lemma 2.6

$$S_{q,v(b+1)+1}^{(b)}(b^{k_1}) \leq \frac{2}{q} \cos \frac{2\pi l}{2l+1} \left(2 \sin \frac{\pi l}{2l+1} \right)^{k_1} + M \gamma^{\frac{k}{s}}$$

and

$$S_{q,v(b+1)+1-b^{k_2}}^{(b)}(b^k) \geq -M \gamma^{\frac{k}{s}}.$$

Therefore

$$\begin{aligned} S_{q,v(b+1)+1}^{(b)}(N) &\leq (1 + o(1)) \left(\frac{2}{q} \cos \frac{2\pi l}{2l+1} \left(2 \sin \frac{\pi l}{2l+1} \right)^{k_1} + \frac{2}{q} \left(2 \sin \frac{\pi l}{2l+1} \right)^{k_1-2} + \frac{2}{q} \left(2 \sin \frac{\pi l}{2l+1} \right)^{k_1-3} + \dots \right) \\ &= (1 + o(1)) \left(\frac{2}{q} \cos \frac{2\pi l}{2l+1} \beta^{k_1} + \frac{2 \beta^{k_1-1}}{q(\beta - 1)} \right) \\ &= (1 + o(1)) \left(\frac{2}{q} \beta^{k_1} \left(-\cos \frac{\pi}{2l+1} + \frac{1}{\beta(\beta - 1)} \right) \right). \end{aligned}$$

Hence (2) of Theorem 1.1 is true for sufficiently large N .

$$S_{q,v(b+1)-1}^{(b)}(N) = S_{q,v(b+1)-1}^{(b)}(b^{k_1}) - S_{q,v(b+1)-1-b^{k_1}}^{(b)}(b^{k_2}) + S_{q,v(b+1)-1-b^{k_1}-b^{k_2}}^{(b)}(b^{k_3}) - \dots$$

From (3) of Lemma 2.6 and (6) of Lemma 2.6

$$S_{q,v(b+1)-1}^{(b)}(b^{k_1}) \leq (1 + o(1)) \frac{2}{q} \left(\cos \frac{2\pi l}{2l+1} - \cos \frac{4\pi l}{2l+1} \right) \left(2 \sin \frac{\pi l}{2l+1} \right)^{k_1-1}$$

and

$$S_{q,v(b+1)-1-b^{k_1}}^{(b)}(b^{k_2}) \geq -M\gamma^{\frac{k}{s}}.$$

Therefore

$$\begin{aligned} S_{q,v(b+1)-1}^{(b)}(N) &\leq (1+o(1)) \left(\frac{2}{q} \left(\cos \frac{2\pi l}{2l+1} - \cos \frac{4\pi l}{2l+1} \right) \left(2 \sin \frac{\pi l}{2l+1} \right)^{k_1-1} + \frac{2}{q} \left(2 \sin \frac{\pi l}{2l+1} \right)^{k_1-2} + \dots \right) \\ &= (1+o(1)) \frac{2}{q} \beta^{k_1-1} \left(\cos \frac{2\pi l}{2l+1} - \cos \frac{4\pi l}{2l+1} + \frac{1}{\beta-1} \right) \leq 0. \end{aligned}$$

Hence (3) of Theorem 1.1 is true. \square

Corollary 2.7. *If b_1 and b_2 are two even numbers and $b_1 > b_2$ then*

$$\lim_{n \rightarrow \infty} \frac{S_{b_2+1,0}^{(b_2)}(b_2(n))}{S_{b_1+1,0}^{(b_1)}(b_1(n))} = 0.$$

Proof. Let $n = \sum_{i=0}^{k-1} \epsilon_i 2^i$ where $\epsilon_i \in \{0, 1\}$. From Lemma 2.6 and Theorem 1.1 one can prove that, for every even b for sufficiently large n there exists constants $c_1 > 0$ and $c_2 > 0$ independent of k such that

$$c_1 \left(2 \sin \frac{\pi b}{2b+2} \right)^k < S_{b+1,0}^{(b)}(b(n)) < c_2 \left(2 \sin \frac{\pi b}{2b+2} \right)^k. \quad (2.5)$$

The Corollary follows from (2.5).

3 Proof of Theorem 1.2

\square

Proof of Theorem 1.2

Proof. If d is even then b is odd and the result is trivial. If $d \geq 3$ is odd and $d|(b+1)$. Let $\phi : A_{d-1} \rightarrow A_b$ be a map defined by

$$\phi((d-1)^{k_1} + \dots + (d-1)^{k_r}) = b^{k_1} + \dots + b^{k_r}$$

for $k_1 > \dots > k_r$. It is easy to see that $n \in A_{d-1} \implies n \equiv \phi(n) \pmod{d}$. Hence

$$S_{d,i}^{(d-1)}((d-1)^{k_1} + \dots + (d-1)^{k_r}) = S_{d,i}^{(b)}(b^{k_1} + \dots + b^{k_r}).$$

From previous theorem for sufficiently large n

$$S_{d,0}^{(d-1)}(n) > 0, S_{d,1}^{(d-1)}(n) < 0 \text{ and } S_{d,-1}^{(d-1)}(n) \leq 0.$$

Hence

$$S_{d,0}^{(b)}(n) > 0, S_{d,1}^{(b)}(n) < 0 \text{ and } S_{d,-1}^{(b)}(n) \leq 0.$$

\square

4 Proof of Theorem 1.3

Proof of Theorem 1.3

Proof. The proof will be similar to proof of Theorem 2 in [2] and Theorem 1.8 in [3]. Note that in this proof b need not be even and b need not equal $2l$. Let s be the order of b in $(\mathbb{Z}/p\mathbb{Z})^*$ and let L_1, \dots, L_t be cosets of $\{1, b, \dots, b^{s-1}\}$. From Lemma 2.2 we have

$$S_{p,0}^{(b)}(b^{4ks-2}) = \frac{1}{p} \sum_{r=1}^t \left(\prod_{j=0}^{s-1} (1 - \zeta_p^{2mb^j}) \right)^{4k} \left(\sum_{l \in L_r} \frac{1}{(1 - \zeta_p^l)(1 - \zeta_p^{lb})} \right),$$

where m is picked from the set L_r . We have

$$\left(\prod_{j=0}^{s-1} (1 - \zeta_p^{2mb^j}) \right)^{4k} = \left(\prod_{i=0}^{s-1} \zeta_p^{mb^i} \right)^{4k} \prod_{i=0}^{s-1} (\zeta_p^{-mb^i} - \zeta_p^{mb^i})^{4k} \geq 0,$$

and

$$\begin{aligned} \operatorname{Re} \left(\sum_{l \in L_r} \frac{1}{(1 - \zeta_p^l)(1 - \zeta_p^{lb})} \right) &= \sum_{l \in L_r} - \frac{\left(\cos \frac{2\pi lb}{2p} \cos \frac{2\pi l}{2p} - \sin \frac{2\pi lb}{2p} \sin \frac{2\pi l}{2p} \right)}{4 \sin \frac{2\pi lb}{2p} \sin \frac{2\pi l}{2p}} \\ &= - \sum_{l \in L_r} \left(-\frac{1}{4} + \frac{1}{4 \tan \frac{2\pi lb}{2p} \tan \frac{2\pi l}{2p}} \right) \\ &= \frac{s}{4} - \sum_{l \in L_r} \frac{1}{4 \tan \frac{\pi lb}{p} \tan \frac{\pi l}{p}}. \end{aligned}$$

Now the following Lemma will help in completing the proof of Theorem 1.3.

Lemma 4.1. *Let L be a coset of $\{1, b, \dots, b^{s-1}\}$ and $p \geq Ct^2(\log p)^2$ then*

$$\sum_{l \in L} \frac{1}{4 \tan \frac{\pi lb}{p} \tan \frac{\pi l}{p}} \geq \frac{C_1(b)p^{\frac{3}{2}}}{t^2 \log p} - C_2(b)s$$

for some positive constants $C_1(b), C_2(b)$ and C which only depend on b .

From Lemma 4.1

$$\operatorname{Re} \left(\sum_{l \in L_r} \frac{1}{(1 - \zeta_p^l)(1 - \zeta_p^{lb})} \right) \leq \frac{-C_1(b)p^{\frac{3}{2}}}{t^2 \log p} + \left(C_2(b) + \frac{1}{4} \right) s$$

Hence if

$$p > \max \left\{ \left(\frac{C_2(b) + \frac{1}{4}}{C_1(b)} \right)^2 (t \log p)^2, C(t \log p)^2 \right\} \leq C'(t \log p)^2$$

then $S_{p,0}^{(b)}(b^{4ks-2}) < 0$. Hence if $p \leq C'(t \log p)^2$ and $t > 1$ one can prove that there exists a constant C^* such that

$$p \leq C^*(t \log t)^2.$$

From the following variation of result of Erdős we can prove the second part of the theorem.

Lemma 4.2. *For every integer $b \geq 2$ and every sequence $\epsilon_p \rightarrow 0$ as $p \rightarrow \infty$ we have*

$$\left| \left\{ p \leq x : \text{ord}_p(b) \leq p^{\frac{1}{2} + \epsilon_p} \right\} \right| = o\left(\frac{x}{\log x}\right).$$

□

Proof of Lemma 4.1

Proof. The residues are taken from $-\frac{p}{2}$ and $\frac{p}{2}$. If we partition L into four sets P_1, P_2, P_3 and P_4 . P_1 contains all $l \in L$ satisfying $|l| \leq \frac{p}{4b}$ and $|b^{-1}l \bmod p| > \frac{p}{4b}$, P_2 contains all l satisfying $|l| \leq \frac{p}{4b}$ and $|b^{-1}l \bmod p| \leq \frac{p}{4b}$, P_3 contains all l satisfying $|l| > \frac{p}{4b}$ and $|bl \bmod p| \leq \frac{p}{4b}$ and P_4 contains all l satisfying $|l| > \frac{p}{4b}$ and $|bl \bmod p| > \frac{p}{4b}$. We have

$$\begin{aligned} \sum := \sum_{l \in L} \frac{1}{4 \tan \frac{\pi lb}{p} \tan \frac{\pi l}{p}} &= \sum_{l \in P_1} \frac{1}{4 \tan \frac{\pi lb}{p} \tan \frac{\pi l}{p}} + \sum_{l \in P_2} \frac{1}{4 \tan \frac{\pi lb}{p} \tan \frac{\pi l}{p}} \\ &+ \sum_{l \in P_3} \frac{1}{4 \tan \frac{\pi lb}{p} \tan \frac{\pi l}{p}} + \sum_{l \in P_4} \frac{1}{4 \tan \frac{\pi lb}{p} \tan \frac{\pi l}{p}}. \end{aligned}$$

Observe that

$$l \in P_1 \Leftrightarrow b^{-1}l \in P_3. \quad (4.6)$$

So

$$\sum_{l \in P_1 \cup P_3} \frac{1}{4 \tan \frac{\pi lb}{p} \tan \frac{\pi l}{p}} = \sum_{l \in P_1} \left(\frac{1}{4 \tan \frac{\pi lb}{p} \tan \frac{\pi l}{p}} + \frac{1}{4 \tan \frac{\pi lb^{-1}}{p} \tan \frac{\pi l}{p}} \right). \quad (4.7)$$

If $|l| \leq \frac{p}{4b}$ then

$$\frac{1}{4 \tan \frac{\pi lb}{p} \tan \frac{\pi l}{p}} = \frac{\cos \frac{\pi lb}{p} \cos \frac{\pi l}{p}}{4 \sin \frac{\pi lb}{p} \sin \frac{\pi l}{p}} \geq \frac{\cos \frac{\pi}{4} \cos \frac{\pi}{4b}}{\frac{4\pi lb}{p} \frac{\pi l}{p}} = c_1(b) \frac{p^2}{l^2}. \quad (4.8)$$

If $|b^{-1}l \bmod p| > \frac{p}{4b}$ then $|\tan \frac{\pi b^{-1}l}{p}| \geq \tan \frac{\pi}{4b}$ and $|\tan \frac{\pi l}{p}| \geq \frac{\pi l}{p}$ which implies

$$\frac{1}{4 \tan \frac{\pi lb^{-1}}{p} \tan \frac{\pi l}{p}} \geq \frac{-1}{\left| 4 \tan \frac{\pi lb^{-1}}{p} \right| \left| \tan \frac{\pi l}{p} \right|} \geq \frac{-p}{4 \tan \frac{\pi}{4b} l} = -\frac{c_2(b)|p|}{|l|}. \quad (4.9)$$

If $|l| > \frac{p}{4b}$ and $|bl \bmod p| > \frac{p}{4b}$ then

$$\frac{1}{4 \tan \frac{\pi lb^{-1}}{p} \tan \frac{\pi l}{p}} \geq \frac{-1}{\tan(\frac{\pi}{4b})^2} = -c_3(b). \quad (4.10)$$

Using (4.6), (4.7), (4.8), (4.9) and (4.10) we have

$$\sum \geq \sum_{l \in P_1} \left(\frac{c_1(b)p^2}{l^2} - \frac{c_2(b)|p|}{|l|} \right) + \sum_{l \in P_2} \frac{c_1(b)p^2}{l^2} + \sum_{l \in P_4} (-c_3(b)) \quad (4.11)$$

$$\geq \sum_{l \in P_1 \cup P_2} \left(\frac{c_1(b)p^2}{l^2} - \frac{c_2(b)|p|}{|l|} \right) + \sum_{l \in P_4} (-c_3(b)). \quad (4.12)$$

Note that

$$\frac{c_1(b)p^2}{l^2} - \frac{c_2(b)|p|}{|l|} \geq -\frac{(c_2(b))^2}{4(c_1(b))^2} = -c_4(b). \quad (4.13)$$

If $\frac{|p|}{|l|} \geq \frac{2c_2(b)}{c_1(b)} = c_5(b)$ then

$$\frac{c_1(b)p^2}{l^2} - \frac{c_2(b)|p|}{|l|} \geq \frac{c_1(b)p^2}{2l^2}. \quad (4.14)$$

From Polya-Vinogradov inequality [8]

$$|\{0 < k \leq 2t\sqrt{p} \log p, k \in L_r\}| > \sqrt{p} \log p.$$

If

$$p \geq 4(c_5(b))^2 t^2 (\log p)^2 = Ct^2 (\log p)^2$$

and $0 < l \leq 2t\sqrt{p} \log p$ then $\frac{|p|}{|l|} \geq c_5(b)$ and from (4.14)

$$\frac{c_1(b)p^2}{l^2} - \frac{c_2(b)|p|}{|l|} \geq \frac{c_1(b)p^2}{2l^2}. \quad (4.15)$$

Hence if $p \geq 4(c_5(b))^2 t^2 (\log p)^2$ and $\frac{p}{2t\sqrt{p} \log p} \leq \frac{p}{4b}$ then from (4.11), (4.13) and (4.15) we have

$$\begin{aligned} \sum &\geq \sum_{\substack{l \in L \\ |l| \leq \frac{p}{4b}}} \left(\frac{c_1(b)p^2}{l^2} - \frac{c_2(b)|p|}{|l|} \right) + \sum_{l \in P_4} (-c_3(b)) \\ &\geq \sum_{\substack{l \in L \\ 0 < l \leq 2t\sqrt{p} \log p}} \left(\frac{c_1(b)p^2}{l^2} - \frac{c_2(b)|p|}{|l|} \right) + \sum_{\substack{l \in L \\ l > 2t\sqrt{p} \log p}} \left(\frac{c_1(b)p^2}{l^2} - \frac{c_2(b)|p|}{|l|} \right) + \sum_{l \in P_4} (-c_3(b)) \\ &\geq \sum_{\substack{l \in L \\ 0 < l \leq 2t\sqrt{p} \log p}} \frac{c_1(b)p^2}{2l^2} + \sum_{\substack{l \in L \\ \frac{|l|}{p} \leq \frac{1}{4b}}} (-c_4(b)) + \sum_{l \in P_4} (-c_3(b)) \\ &\geq \frac{c_1(b)p^{\frac{3}{2}}}{8t^2 \log p} - c_4(b)s - c_3(b)s = \frac{C_1(b)p^{\frac{3}{2}}}{t^2 \log p} - C_2(b)s. \end{aligned}$$

□

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